A Dual Characterization of Banach Spaces not Containing ℓ^1

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A Dual Characterization of Banach Spaces not Containing ℓ^1

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Summary. It is shown that a Banach space E does not contain ℓ^1 iff certain subsets of E^* (the dual space of E) are relatively compact. This result has some consequences concerning Dunford-Pettis and limited sets in E; moreover, a result concerning operators defined on Banach spaces with the Dunford-Pettis property follows from the main result.

1. Introduction. Let E be a Banach space having a dual space denoted by E^*. If a bounded subset K of E^* is such that for any weakly null sequence $(x_n) \subset E$ one has $\lim_n \sup_K |x_n(x^*)| = 0$, we shall say that it is an (L) set ([7]).

The main purpose of our note is to characterize Banach spaces not containing ℓ^1 using (L) sets. This result will have several corollaries concerning Dunford-Pettis and limited sets ([1], [2]); we recall that a subset K of a Banach space is called a *Dunford-Pettis set* (resp. a *limited set*) iff for any weakly null (resp. weak* null) sequence $(x_n^*) \subset E^*$ one has $\lim_n \sup_K |x_n^*(x)| = 0$. Other corollaries (partially known) related to the Dunford-Pettis property and Gelfand-Phillips spaces will follow; we recall that a Banach space E verifies the *Dunford-Pettis property* iff any its relatively weakly compact subset is a Dunford-Pettis set and it is a *Gelfand-Phillips space* iff any its limited subset is relatively compact ([1], [4]). At the end, we derive a result concerning operators defined on a Banach space with the Dunford-Pettis property; also this result has in turn some consequences.

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2. Results. Our main result uses the following characterization of Banach spaces not containing ℓ^1 due to Odell (see [14], p. 377).

`Theorem 1.` *A Banach space E does not contain ℓ^1 iff any completely continuous operator from E to any Banach space F is compact (Here completely continuous signifies that it maps weakly null sequences into norm null sequences; recently many authors call such an operator a Dunford-Pettis operator).*

The central theorem of the paper is the following

`Theorem 2.` *A Banach space E does not contain ℓ^1 iff any (L) subset of E is relatively compact.*

Proof. K be an (L) set in E^*. Following [7] we define an operator $T : E \to B(K)$ ($B(K)$ is the usual space of all bounded scalar functions on K) by putting $T(x)(x^*) = x^*(x)$. We can see easily that T is completely continuous and hence it is compact, from Theorem 1. Also, T^* is compact. Now, let $x^* \in K$; we define F_{x^*} in $(B(K))^*$ by putting $F_{x^*}(f) = f(x^*)$ for any $f \in B(K)$; this implies that $T^*(F_{x^*})(x^*) = x^*(x)$ and hence $T^*(F_{x^*}) = x^*$ for any $x^* \in K$. Since $K = \{T^*(F_{x^*}) : x^* \in K\} \subset T^*(B_{(B(K))^*})$ we have that K is relatively compact. Conversely, let $T : E \to F$ be an arbitrary completely continuous operator; by taking as K the subset $T^*(B_{F^*})$ we easily show that T^* (and hence T) is compact. The proof is complete.

`Remark 1.` It is known ([6]) that a Banach space E does not contain ℓ^1 iff E^* satisfies the weak Radon-Nikodym property. Hence, Theorem 2 gives a characterization of dual Banach spaces with the weak Radon-Nikodym property. Sometimes we shall use this equivalence without any advice.

Theorem 2 has some corollaries which we now prove. The first is related to Dunford-Pettis sets

`Corollary 1.` *Let F be a closed subspace of a dual Banach space E^* with E not containing ℓ^1. Then, any Dunford-Pettis set in F is relatively compact.*

Proof. Let K be a Dunford-Pettis set in F; obviously, it is even a Dunford-Pettis set in E^*. We consider a weakly null sequence (x_n) in E; if we think of it as of a sequence in E^{**} we easily see that it converges weakly to θ. The definition of a Dunford-Pettis set says that K is an (L) set in E^*. An appeal to Theorem 2 concludes the proof.

Corollary 1 has an immediate consequence; thus the proof is omitted.

`Corollary 2.` *Let F be a Banach space in which there exists a Dunford-Pettis set which is not relatively compact. then, if $F \subset E^*$, E has to contain ℓ^1.*

Corollary 2 improves the following well-known result (see [5], p. 213).

`Corollary 3.` *If F is a Banach space with the Dunford-Pettis property but without the Schur property and $F \subset E^*$, then E contains ℓ^1.*

The following is another consequence of Corollary 1.

`Corollary 4.` *Let E and F be Banach spaces. We suppose that F is a closed subspace of a dual Banach space Z^* with a predual Z not containing ℓ^1. Then, any completely continuous operator $T : F^* \to E^*$ is compact, provided that $T = P^*$, for some operator $P : E \to F$.*

Proof. Easily we can see that if $T = P^*, P : E \to F$ then T is completely continuous iff P maps bounded sets into Dunford-Pettis sets; Corollary 1 gives that P is compact. The proof is complete.

Now, we use Corollary 1 in order to indicate a class of Gelfand-Phillips spaces; these Banach spaces have been recently investigated in [2] and in a paper (still unpublished as far as we know) cited in [4], p. 150; they are important in connection with the compactness of the range of Pettis integrals (see [4], p. 150).

`Corollary 5.` *Let F be a closed subspace of a dual Banach space E^* with E not containing ℓ^1. Then, F is a Gelfand-Phillips space.*

Proof. From the definitions of Dunford-Pettis sets and limited sets it follows that any limited set is a Dunford-Pettis set. Hence, Corollary 1 concludes the proof.

This corollary improves a result cited in [4], p. 150, which only affirms that dual Banach spaces of spaces not containing ℓ^1 are Gelfand-Phillips spaces. Moreover, we observe that the class of Gelfand-Phillips spaces obtained here is different from those indicated in [2], i.e. Banach spaces with the Schur property, separably complemented Banach spaces and Banach spaces having weak* sequentially compact dual balls; indeed, from the results of [8] follows the existence of dual Banach spaces having the weak Radon-Nikodym property, but not separably complemented or having the Schur property. Further, we observe that dual Banach spaces E* with weak* sequentially compact dual balls have the weak Radon-Nikodym property; indeed, if (x_n) is a bounded sequence in $E \subset E^{**}$, there is a subsequence $(x_{k(n)})$ converging weak* in E^{**}; hence, $(x_{k(n)})$ is a weak Cauchy sequence in E; an appeal to well-known results (see [5], for example, and [6]) concludes our proof; this implication cannot be reversed, as it is observed in [15]. We also observe that Corollary 5 is a partial improvement of a result of Musial ([9]) about the range of certain countably additive measures.

The following corollary concerns with operators defined on Banach spaces having the Dunford-Pettis property.

`Corollary 6.` *Let F a Banach space having the Dunford-Pettis property and let E^* be a dual Banach space with E not containing ℓ^1. Then, any operator $T : F \to E^*$ is completely continuous.*

Proof. Let (y_n) be a weakly null sequence in F. We first prove that $(T(y_n))$ is a Dunford-Pettis set in E^*; an appeal to Corollary 1 will finish the proof. Let $(x_n^{**}) \subset E^{**}$ be a weakly null sequence; we have $T(y_n)(x_n^{**}) = y_n(T^*(x_n^{**}))$, for any $n \in N$; since $T(x_n^{**}) \xrightarrow{w} 0$ in F^* the Dunford-Pettis property of F implies that $T(y_n)(x_n^{**}) \to 0$ (we are using a characterization of the Dunford-Pettis property which is due to Grothendieck; see [3] for example). This easily gives that $(T(y_n))$ is a Dunford-Pettis set.

The corollary is a generalization of a result of [12] which in turn improves a theorem of [13]. Moreover, it allows us to enlarge a characterization of Banach spaces not containing ℓ^1 due to Pelczynski ([11]).

`Corollary 7.` *The following facts are equivalent*

i) E does not contain ℓ^1*,*

ii) any operator from a Banach space F having the Dunford-Pettis property to E^* *is completely continuous,*

iii) the same as ii) with $F = L^1([0,1])$*.*

Proof. i) implies ii) is Corollary 6; ii) implies iii) is obvious; iii) implies i) is in [11].

`Corollary 8.` *Let* (S, Σ, μ) *be a finite measure space and let F be a Banach space such that* F^* *has the Schur property. Assume that* μ *is not purely atomic. Then,* $L^1(\mu,F)$ *cannot be isomorphic to a subspace of a weakly compactly generated dual Banach space.*

Proof. In [1] it is shown that $L^1(\mu,F)$ has the Dunford-Pettis property. On the other hand a weakly compactly generated dual space has weak* sequentially compact dual balls (see [5]) and we have already observed that such a space has the weak Radon-Nikodym property. Corollary 6 concludes our proof.

This improves a well-known result of Gelfand-Pelczynski (see Corollary 9 of p. 83 of [3]); also Corollary 10 of p. 83 of [3] can be obtained as a corollary of Corollary 6.

Other consequences are the following results

`Corollary 9.` *Let F be a Banach space having the Dunford-Pettis and the Reciprocal Dunford-Pettis property (see [7]). Any operator* $T : F \to E^*$*, with E not containing* ℓ^1*, is weakly compact.*

Proof. Corollary 6 says that T is completely continuous. The definition of the Reciprocal Dunford-Pettis property gives that T is weakly compact.

`Corollary 10.` *Let* (S, Σ, μ) *be a finite measure space and let E be a Banach space not containing* ℓ^1*. Any operator* $T : E \to L^1(\mu)$ *is strictly cosingular (for the definition of strict cosingularity we refer the reader to [10]).*

Proof. $T : L^\infty(\mu) \to E^*$ is weakly compact from Corollary 9; indeed, the space $L^\infty(\mu)$ as a $C(H)-$space has both the Dunford-Pettis ([3]) and the Reciprocal Dunford-Pettis property ([7]). Thus T is weakly compact. Since $L^1(\mu)$ has the Dunford-Pettis property

([3]), a result of Pelczynski (see [10]) concludes our proof.

Added in proof. The referee has observed that Corollary 7 was obtained by Fakhoury in the paper *Sur les espaces de Banach ne contenant pas* $\ell^1(N)$, Math. Scand., 41 (1977), 277-289, in the identical wording.

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